JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **147**, 81–96 (1990)

Stability Theorems of Stochastic Difference Equations

TAKESHI TANIGUCHI

Department of Mathematics, Kurume University,
Miimachi, Kurume, Fukuoka, Japan

Submitted by E. Stanley Lee

Received September 29, 1988

In this paper we prove the general comparison theorem for the difference inequalities and several stochastic stability theorems of the nonlinear difference equations using this theorem.

Our theorems generalize Ma and Caughey's theorems.

1. INTRODUCTION

The qualitative properties of solutions of ordinary differential equations have been studied by means of Liapunov's second method by many authors [6]. Among such investigations the comparison theorem has been one of the most powerful theorems.

Now, as is well known, the stability problem of solutions of difference equations is a very important one for analyzing so called sampled data systems, and for studying this stability problem the discrete comparison theorem is also prominently useful [7]. Furthermore the data analysis in mechanical, electrical, and control engineering, and the other practical problems have necessitated a study of stochastic difference equations.

Recently Ma and Caughey presented several stability theorems of solutions of non-linear stochastic difference equations [8–11].

In this paper we prove the general discrete comparison theorem which is a generalization of Ma and Caughey's comparison theorem and using this theorem investigate several stability theorems of non-linear stochastic difference equations. From our theorems Ma and Caughey's theorems [11] are induced simply and easily as corollaries.

2. PRELIMINARIES

Let Ω be a probability space and let $A_k(\omega)$, $\omega \in \Omega$ $(k = 0, 1, 2, \ldots)$ be independent $n \times n$ stochastic matrices, respectively. Suppose that an initial

0022-247X/90 $3.00

R^n-valued probability distribution x_0 defined on Ω is independent of $\{A_k(\omega)\}$ respectively. N_0 denotes the set of non-negative integers, R^+ denotes the set of non-negative real numbers.

Consider the following linear stochastic difference equation and the perturbed stochastic difference equation,

$$x_{k+1} = A_k(\omega)\, x_k, \qquad k \in N_0 \tag{1}$$

$$x_{k+1} = A_k(\omega)\, x_k + f(k, x_k), \qquad k \in N_0, \tag{2}$$

where $x_k \in R^n$ and f is a continuous R^n-valued function defined on $N_0 \times R^n$ with $f(k, 0) = 0$ for all $k \in N_0$. Now we give several stability definitions of the solutions of (1). Throughout this paper a number p denotes a positive real number.

DEFINITION 1. The solutions of (1) are said to be

(SS_1) strongly pth mean stable with the order C if there exists a constant $C \geqslant 1$ such that

$$E(\|A_{m-1} \cdots A_n\|^p) \leqslant C \qquad \text{for all} \quad m > n \geqslant 0,$$

(SS_2) strongly pth mean quasi-asymptotically stable if

$$\lim_{m \to \infty} E(\|A_{m-1} \cdots A_n\|^p) = 0 \qquad \text{for all} \quad m > n \geqslant 0.$$

DEFINITION 2. The solutions of (1) are said to approach zero fast with the order (C, δ) if there exist a constant $C \geqslant 1$ and $0 < \delta < 1$ such that

$$E(\|A_{m-1} \cdots A_n\|) \leqslant C\delta^{m-n+1} \qquad \text{for all} \quad m > n \geqslant 0.$$

DEFINITION 3. The solutions of (1) are said to approach zero fast with the order (K, p, α) if there exists a constant $K \geqslant (\frac{1}{2})^p$ such that

$$E(\|A_{m-1} \cdots A_n\|^p) \leqslant K(n+1)^p/m^p \qquad \text{for all} \quad m > n \geqslant 0.$$

DEFINITION 4. The solutions of (1) are said to approach zero fast with the order (K, p, δ) if there exists a constant $K \geqslant (\frac{1}{2})^p$ and $0 < \delta < 1$ such that

$$E(\|A_{m-1} \cdots A_n\|^p) \leqslant K(1/m)^p\, \delta^{m-n+1} \qquad \text{for all} \quad m > n \geqslant 0.$$

DEFINITION 5. The solutions of (1) are said to approach zero fast with the order (β, t, p) if there exists positive sequences $\{\beta_n\}$ and $\{t_n\}$ such that $(n+1)^p \beta_n \to 0$ as $n \to \infty$ and

$$E(\|A_{m-1} \cdots A_n\|^p) \leqslant \beta_m t_n.$$

Next we give the stochastic stability definitions for the zero solution of (2).

DEFINITION 6. The zero solution of (2) is said to be

(S_1) pth mean stable for $p>0$ if for any $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that

$$E(\|x_k\|^p)<\varepsilon \qquad \text{for all} \quad k\in N_0$$

provided $E(\|x_0\|^p)<\delta(\varepsilon)$,

(S_2) pth mean quasi-asymptotically stable for $p>0$ if there exists a $\delta_0>0$ such that

$$E(\|x_k\|^p)\to 0 \qquad \text{as} \quad k\to\infty$$

provided $E(\|x_0\|^p)<\delta_0$,

(S_3) pth mean asymptotically stable for $p>0$ if it is pth mean stable and pth mean quasi-asymptotically stable.

3. LEMMAS

In this section several lemmas are given which we use in the proof of the theorems. See [16] for the proof of Lemma 1. Lemma 2 is due to MA[11] essentially. And Lemma 3 is a main lemma.

LEMMA 1. *If $a_i\geqslant 0$ for all non-negative integers i, then*

$$\prod_{i=0}^{n}(1+a_i) \qquad \textit{and} \qquad \sum_{i=0}^{n} a_i$$

converge or diverge together.

LEMMA 2. *If a sequence $\{a_j\}$ converges to 1 and $0<\alpha<1$, then*

$$\lim_{n\to\infty}\prod_{j=1}^{n}[(a_j)^p\,\alpha]=0.$$

Proof. Take a $\beta>0$ such that $1<\beta<1/(\alpha)^{1/p}$. Then there exists a positive integer N such that if $n\geqslant N$, then $0<a_n<\beta$ since $a_n\to 1$ as $n\to\infty$. Thus $|\prod_{j=1}^{n}[(a_j)^p\,\alpha]|<|\prod_{j=1}^{N}[(a_j)^p\,\alpha]|\prod_{j=N+1}^{n}[\beta^p\alpha]$ for all $n>N$, which completes the proof because of $(\beta^p\alpha)^{n-N}\to 0$ as $n\to\infty$.

Next we present the general discrete comparison theorem which plays a fundamental role in this paper.

LEMMA 3. *Let $F(n, s, u): N_0 \times N_0 \times R^+ \to R^+$ be a monotone non-decreasing continuous function in u and let two positive sequences $\{v_n\}$ and $\{x_n\}$ satisfy the following conditions respectively*:

$$v_n \geqslant \left[\sum_{s=0}^{n-1} F(n, s, v_s) + p_n\right], \qquad p_n \geqslant 0$$

and

$$x_n \leqslant \rho(n) \left[\sum_{s=0}^{n-1} F(n, s, x_s) + p_n\right], \qquad p_n \geqslant 0.$$

Suppose that the following condition holds for all $n \in N_0$ $(n > s)$:

(a) $0 < \rho(n) \leqslant 1$ *and* $F(n, s, u) \leqslant F(n, s, \alpha u)$, $\alpha \geqslant 1$, *or*

(b) $\rho(n) \geqslant 1$ *and* $\alpha F(n, s, u) \leqslant F(n, s, \beta u)$, $0 \leqslant \alpha \leqslant \beta \leqslant 1$.

If (a) *holds, then $x_n \leqslant \rho(n)\, v_n$ for each $n \in N_0$ provided $x_0 \leqslant \rho(0)\, v_0$.*
If (b) *holds, then $x_n \leqslant \prod_{i=0}^{n} \rho(i)\, v_n$ for each $n \in N_0$ provided $x_0 \leqslant \rho(0)\, v_0$.*

Proof. If (a) holds, then we obtain that for all $n \in N_0$

$$x_n/\rho(n) - \left[\sum_{s=0}^{n-1} F(n, s, x_s/\rho(s))\right] \leqslant v_n - \sum_{s=0}^{n-1} F(n, s, v_s),$$

which implies that $x_n \leqslant \rho(n)\, v_n$ for each $n \in N_0$ provided $x_0 \leqslant \rho(0)\, v_0$. If (b) holds, then set $a(s) = \prod_{i=0}^{s} \rho(i)$. We obtain that for all $n \in N_0$

$$x_n/a(n) - \left[\sum_{s=0}^{n-1} F(n, s, x_s/a(s))\right] \leqslant v_n - \sum_{s=0}^{n-1} F(n, s, v_s),$$

which implies that $x_n \leqslant a(n)\, v_n$ for each $n \in N_0$ provided $x_0 \leqslant \rho(0)\, v_0$. This completes the proof.

We remark that Lemma 3 generalizes Lemma 3.1 in [11].

DEFINITION 7. The function $F(n, u): N_0 \times R^+ \to R^+$ is said to have

$$\text{condition } (\alpha) \text{ if } \alpha F(n, u) \leqslant F(n, \alpha u) \qquad \text{for} \quad \alpha \geqslant 1$$

and

$$\text{condition } (\beta) \text{ if } \alpha F(n, u) \leqslant F(n, \beta u) \qquad \text{for} \quad 0 \leqslant \alpha \leqslant \beta \leqslant 1.$$

4. Stochastic Stability of Non-linear Difference Equations

In this section we prove stochastic stability theorems of non-linear stochastic difference equations.

Let $\Delta: R^1 \to R^1$ be the difference operator defined by

$$\Delta w_n = w_{n+1} - w_n.$$

In general, the stability definition of the zero solution of the difference equation $\Delta v_n = F(n, v_n)$ is given for the function F such that the domain and the range are $N_0 \times R^1$ and R^1, respectively [7].

However the non-negativity of variables and values of F is required as a fundamental condition in economical, physical, and control systems. Therefore we have the following definition of stability of the zero solution of the difference equation for the function F with the non-negative domain and range.

Definition 8. The zero solution of the difference equation $\Delta v_n = F(n, v_n)$, $F(n, 0) = 0$, where $F(n, u): N_0 \times R^+ \to R^+$, is said to be stable if for any $\alpha > 0$, there exists a $\delta(\alpha) > 0$ such that for all $n \in N_0$

$$0 < v_n < \varepsilon$$

provided $0 < v_0 < \delta(\varepsilon)$.

Asymptotic stability of the zero solution is defined by the same way as (S_3) in Definition 6.

To begin with, the first mean stability theorem of the difference equation (2) is proved.

Theorem 1. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(1a) *the solutions of* (1) *are strongly first mean stable with the order* C,

(1b) $E(\|f(n, x)\|) \leqslant F(n, E(\|x\|))$, $F(n, 0) = 0$ *for all* $n \in N_0$, *where* $x: \Omega \to R^n$ *is a random variable and* $F(n, u): N_0 \times R^+ \to R^+$ *is a monotone non-decreasing continuous function in* u *for each fixed* $n \in N_0$,

(1c) *the zero solution of the difference equation*

$$\Delta v_n = CF(n, v_n) \tag{3}$$

is (*asymptotically*) *stable.*

Then the zero solution of (2) *is first mean* (*asymptotically*) *stable.*

Proof. The solution of (2) is of the form

$$x_n = \prod_{j=0}^{n-1} A_j x_0 + \sum_{s=0}^{n-1} \left(\prod_{j=s+1}^{n-1} A_j \right) f(s, x_s) \qquad \text{for all} \quad n \in N_0. \tag{4}$$

Since the initial random variable x_0 is independent of each $A_k(\omega)$ $(k \in N_0)$ and condition (1a) holds, we obtain that

$$E(\|x_n\|) \leqslant CE(\|x_0\|) + \sum_{s=0}^{n-1} CF(s, E(\|x_s\|)) \qquad \text{for all} \quad n \in N_0$$

using condition (1b). Let v_n be the solution of (3) with the initial value $v_0 = CE(\|x_0\|)$. Then applying Lemma 3 with $\rho(n) \equiv 1$, for all $n \in N_0$,

$$E(\|x_n\|) \leqslant v_n.$$

Hence by condition (1c), for any $\varepsilon > 0$ there exists a $\delta_1(\varepsilon)$ such that if $0 < v_0 < \delta_1(\varepsilon)$, then $0 < v_n < \varepsilon$. Therefore setting $\delta(\varepsilon) = \delta_1(\varepsilon)/C$, we obtain that if $E(\|x_0\|) < \delta(\varepsilon)$, then $E(\|x_n\|) < \varepsilon$, which implies that the zero solution of (2) is first mean stable. The proof of the theorem is complete.

Theorem 3.2 in [11] is obtained as the corollary of this theorem. We show this.

COROLLARY 1. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(1d) *the solutions of* (1) *are strongly first mean stable with the order C,*

(1e) $E(\|f(n, x)\|) \leqslant B(n)\, E(\|x\|)$, $B(n) \geqslant 0$ *for all* $n \in N_0$.

If the series $\sum_{i=0}^{n} B(i)$ *converges, then the zero solution of* (2) *is first mean stable.*

Proof. Set $F(n, u) = B(n)\, u$, $u \in R^+$ for each $n \in N_0$. Then the solution of the difference equation $\Delta v_n = CF(n, v_n)$ is of the form

$$v_n = v_0 + \sum_{s=0}^{n-1} CB(s)\, v_s.$$

Therefore, as is well known, we obtain that

$$v_n = v_0 \prod_{s=0}^{n-1} (1 + CB(s)),$$

which implies that the zero solution of $\Delta v_n = CF(n, v_n)$ is stable by

Lemma 1. Since all conditions of Theorem 1 are satisfied, the proof of Corollary 1 is complete.

Now we shall prove the pth mean stability theorem of the stochastic difference equation (2).

THEOREM 2. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(2a) *the solutions of* (1) *approach zero fast with the order* (K, p, α),

(2b) $E(\|f(n, x)\|^p) \leqslant F(n, E(\|x\|^p))$, $\sum_{s=0}^{n-1} (s+2)^p F(s, u_s) \leqslant \sum_{s=0}^{n-1} G(s, u_s)$ *for all* $n \in N_0$ *where* $x: \Omega \to R^n$ *is a random variable,* $F(n, u), G(n, u): N_0 \times R^+ \to R^+$ *are monotone non-decreasing continuous functions in* u, $G(n, 0) = 0$ *for each fixed* $n \in N_0$ *and* $u_s \in R^+$,

(2c) *the zero solution of the difference equation*

$$\Delta v_n = 2^p K G(n, v_n)$$

is (asymptotically) stable.

Then the zero solution of (2) *is* pth *mean (asymptotically) stable.*

Proof. Since the solutions of (1) approach zero fast with the order (K, p, α), by (4) and condition (2b) we obtain that

$$\begin{aligned} E(\|x_n\|^p) &\leqslant E\left(\left\|\prod_{j=0}^{n-1} A_j\right\| \|x_0\| + \sum_{s=0}^{n-1} \left\|\left(\prod_{j=s+1}^{n-1} A_j\right)\right\| \|f(s, x_s)\|\right)^p \\ &\leqslant 2^p K E(\|x_0\|^p) + 2^p \sum_{s=0}^{n-1} n^p K[(s+2)/n]^p F(s, E(\|x_s\|^p)) \\ &\leqslant 2^p K E(\|x_0\|^p) + \sum_{s=0}^{n-1} 2^p K G(s, E(\|x_s\|^p)). \end{aligned}$$

Hence by Lemma 3 and condition (2c) we obtain that for any $\varepsilon > 0$ setting $v_0 = 2^p K E(\|x_0\|^p)$, for all $n \in N_0$

$$\begin{aligned} E(\|x_n\|^p) &\leqslant v_n \\ &< \varepsilon \end{aligned}$$

provided $E(\|x_0\|^p) < \delta(\varepsilon)/2^p K$, where $\delta(\varepsilon)$ is a positive number such that if $0 < v_0 < \delta(\varepsilon)$, then $0 < v_n < \varepsilon$ for all $n \in N_0$, the existence of which is guaranteed because of condition (2c). This implies that the zero solution of (2) is pth mean stable. Thus the proof of the theorem is complete.

As the corollary, we get a result similar to Theorem 3.6 in [11].

COROLLARY 2. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(2d) *the solutions of* (1) *approach zero fast with the order* (K, p, α),

(2e) $E(\|f(n, x)\|^p) \leqslant B(n)\, E(\|x\|^p)$ *for all* $n \in N_0$,

(2f) $B(n) \leqslant \beta_n t_n$, $t_n \geqslant 0$, *where* $s(n) = \sum_{s=0}^{n-1} (s+2)^p \beta_s \leqslant B$ (*a constant*) *for all* $n \in N_0$ *and the series* $\sum_{i=0}^{n-1} t_i$ *converges.*

Then the zero solution of (2) *is pth mean stable.*

Proof. Set $F(n, u) = B(n)\, u$, $G(n, u) = Bt_n u$, $u \in R^+$ for each $n \in N_0$. We show that the zero solution of the difference equation $\Delta v_n = 2^p KBt_n v_n$ is stable. Since for all $n \in N_0$

$$v_n = v_0 \prod_{s=0}^{n-1} (1 + 2^p KBt_s)$$

and $\sum_{s=0}^{\infty} t_s < \infty$, there exists a $T > 0$ such that $\prod_{s=0}^{\infty} (1 + 2^p KBt_s) < T < \infty$ by Lemma 1, which implies that the zero solution is stable. Thus the proof is complete since all conditions of Theorem 2 hold.

Next we consider the quasi-asymptotic stability theorems.

THEOREM 3. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(3a) *the solutions of* (1) *approach zero fast with the order* (C, δ),

(3b) $E(\|f(n, x)\|) \leqslant F(n, E(\|x\|))$ *for all* $n \in N_0$, *where* $x: \Omega \to R^n$ *is a random variable and* $F(n, u): N_0 \times R^+ \to R^+$ *is a monotone non-decreasing continuous function in u for each fixed* $n \in N_0$ *and moreover F has condition* (α),

(3c) $\delta^n v_n \to 0$ *as* $n \to \infty$, *where* v_n *is the solution of the difference equation* $\Delta v_n = CF(n, v_n)$ *with the initial value* $v_0 = CE(\|x_0\|)$.

Then the zero solution of (2) *is first mean quasi-asymptotically stable.*

Proof. By condition (3a) and (4), we obtain that for all $n \in N_0$

$$E(\|x_n\|) \leqslant C\delta^n E(\|x_0\|) + \sum_{s=0}^{n-1} C\delta^{n-s} F(s, E(\|x_s\|)).$$

Hence it follows that since $0 < \delta < 1$ and F has condition (α),

$$E(\|x_n\|/\delta^n) \leqslant CE(\|x_0\|) + \sum_{s=0}^{n-1} CF(s, E(\|x_s\|)/\delta^s).$$

Therefore by Lemma 3, $E(\|x_n\|/\delta^n) \leqslant v_n$, which implies that $E(\|x_n\|) \leqslant \delta^n v_n$

and hence $E(\|x_n\|) \to 0$ as $n \to \infty$ by condition (3c). The proof of the theorem is complete.

As the corollary of this theorem we obtain Theorem 3.1 in [11].

COROLLARY 3. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(3d) *the solutions of* (1) *approach zero fast with the order* (C, δ),

(3e) *there exists a sufficiently small constant* $L > 0$ *such that*

$$E(\|f(n, x)\|) \leqslant LE(\|x\|) \qquad \textit{for all} \quad n \in N_0.$$

Then the zero solution of (2) *is first mean quasi-asymptotically stable.*

Proof. Set $F(n, u) = Lu$, $u \in R^+$ for each $n \in N_0$. Since condition (3b) in Theorem 3 is satisfied, we shall show that condition (3c) holds. Let v_n be a solution of the difference equation $\Delta v_n = CLv_n$. Then

$$v_n = (1 + CL)^n v_0.$$

Thus we obtain that $\delta^n v_n = (\delta + CL\delta)^n v_0$. Since L is sufficiently small, we may assume $\delta + CL\delta < 1$. Therefore condition (3c) of Theorem 3 is satisfied, which completes the proof.

THEOREM 4. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(4a) *the solutions of* (1) *approach zero fast with the order* (K, p, δ),

(4b) $E(\|f(n, x)\|^p) \leqslant F(n, E(\|x\|^p))$ *for all* $n \in N_0$, *where* $x: \Omega \to R^n$ *is a random variable,* $F(n, u): N_0 \times R^+ \to R^+$ *is a monotone non-decreasing continuous function in* u *for each fixed* $n \in N_0$, *and moreover the function* $F(n, u)$ *has condition* (α),

(4c) $\delta^n v_n \to 0$ *as* $n \to \infty$, *where* v_n *is the solution of the difference equation* $\Delta v_n = 2^p KF(n, v_n)$ *with the initial value* $v_0 = 2^p KE(\|x_0\|^p)$.

Then the zero solution of (2) *is pth mean quasi-asymptotically stable.*

Proof. By (4) and condition (4a), we obtain that for all $n \in N_0$

$$\begin{aligned} E(\|x_n\|^p) &\leqslant E\left(\left\|\prod_{j=0}^{n-1} A_j\right\| \|x_0\| + \sum_{s=0}^{n-1} \left\|\prod_{j=s+1}^{n-1} A_j\right\| \|f(s, x_s)\|\right)^p \\ &\leqslant (n+1)^p \left(K(1/n)^p \, \delta^n E(\|x_0\|^p)\right. \\ &\qquad \left. + \sum_{s=0}^{n-1} K(1/n)^p \, \delta^{n-s} E(\|f(s, x_s)\|^p)\right). \end{aligned}$$

Hence it follows that

$$E(\|x_n\|^p)/\delta^n \leqslant 2^p K E(\|x_0\|^p) + \sum_{s=0}^{n-1} 2^p K F(s, E(\|x_s\|^p)/\delta^s).$$

Therefore by Lemma 3, $E(\|x_n\|^p)/\delta^n \leqslant v_n$, which implies that $E(\|x_n\|^p) \to 0$ as $n \to \infty$ by condition (4c). This completes the proof of the theorem.

Similarly we obtain the following theorem.

THEOREM 4′. *Suppose that conditions* (4a), (4b) *hold and the function* F *has condition* (β). *Furthermore setting* $\rho(i) = (1 + 1/i)^p$, $i \geqslant 1$, $\rho(0) = 1$,

(4c′) $\delta^n \prod_{i=0}^{n} \rho(i)\, v_n \to 0$ *as* $n \to \infty$,

where v_n *is the solution of the difference equation* $\Delta v_n = KF(n, v_n)$ *with the initial value* $v_0 = KE(\|x_0\|^p)$ *and* $K \geqslant 1$.

Then the zero solution of (2) *is pth mean quasi-asymptotically stable.*

As the corollary of Theorem 4′ we obtain Theorem 3.3 in [11]. Of course by Theorem 4 we can also induce Theorem 3.3. However, note that although we can prove Theorem 3.3 (Corollary 4) by using Theorem 4′ if $0 < L < (1 - \delta)/(K\delta)$, we can not apply Theorem 4 for proving it if $L > (1 - \delta)/(2^p K\delta)$ where L satisfies condition (4e).

COROLLARY 4. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(4d) *the solutions of* (1) *approach zero fast with the order* (K, p, δ) $(K \geqslant 1)$,

(4e) *there exists a sufficiently small constant* $L > 0$ *such that*

$$E(\|f(n, x)\|^p) \leqslant LE(\|x\|^p) \qquad \text{for all} \quad n \in N_0.$$

Then the zero solution of (2) *is pth mean quasi-asymptotically stable.*

Proof. Set $F(n, u) = Lu$, $u \in R^+$ for each $n \in N_0$. We shall show that condition (4c′) in Theorem 4′ holds. Let v_n be a solution of the difference equation $\Delta v_n = KLv_n$. Then, $v_n = (1 + KL)^n v_0$. Thus we obtain that

$$\delta^n \prod_{i=0}^{n} \rho(i)\, v_n = (\delta + KL\delta)^n v_0 \prod_{i=0}^{n} \rho(i).$$

Since L is sufficiently small, we may assume that $\delta + \delta KL < 1$. Therefore by Lemma 2, condition (4c′) holds, which completes the proof.

Next we prove another pth mean quasi-asymptotic stability theorem.

THEOREM 5. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(5a) *the solutions of* (1) *approach zero fast with the order* (β, t, p),

(5b) $E(\|f(n, x)\|^p) \leqslant F(n, E(\|x\|^p))$ *for all* $n \in N_0$, *where* $x: \Omega \to R^n$ *is a random variable and* $F(n, u): N_0 \times R^+ \to R^+$ *is a monotone non-decreasing continuous function in u for each fixed* $n \in N_0$,

(5c) $(n+1)^p \beta_n v_n \to 0$ *as* $n \to \infty$, *where an* $R \geqslant 1$ *is a constant such that* $R > (n+1)^p \beta_n$ *for all* $n \in N_0$, $t_0 R \geqslant 1$, *and* v_n *is the solution of the difference equation* $\Delta v_n = R t_{n+1} F(n, v_n)$ *with the initial value* v_0 $(= t_0 R E(\|x_0\|^p))$.

Then the zero solution of (2) *is pth mean quasi-asymptotically stable.*

Proof. By (4) and the condition of the theorem we obtain that for all $n \in N_0$

$$E(\|x_n\|^p) \leqslant E\left(\left\|\prod_{j=0}^{n-1} A_j\right\| \|x_0\| + \sum_{s=0}^{n-1} \left\|\prod_{j=s+1}^{n-1} A_j\right\| \|f(s, x_s)\|\right)^p$$
$$\leqslant (n+1)^p \left[\beta_n t_0 E(\|x_0\|^p) + \sum_{s=0}^{n-1} \beta_n t_{s+1} F(s, E(\|x_s\|^p))\right]$$
$$\leqslant \rho(n) \left[t_0 R E(\|x_0\|^p) + \sum_{s=0}^{n-1} t_{s+1} R F(s, E(\|x_s\|^p))\right],$$

where $\rho(n) = (n+1)^p \beta_n / R$ for $n \geqslant 1$ and $\rho(0) = 1$. Therefore by Lemma 3, noting that $0 < \rho(n) \leqslant 1$ for all $n \in N_0$,

$$E(\|x_n\|^p) \leqslant \rho(n)\, v_n,$$

which implies that $E(\|x_n\|^p) \to 0$ as $n \to \infty$ since $\rho(n)\, v_n \to 0$ as $n \to \infty$. This completes the proof.

As the corollary of this theorem we obtain Theorem 3.4 in [11].

COROLLARY 5. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(5d) *the solutions of* (1) *approach zero fast with the order* (β, t, p),

(5e) *there exist two positive constants* L_1 *and* L_2 *such that*

$$E(\|f(n, x)\|^p) \leqslant L_1 + L_2 E(\|x\|^p) \qquad \text{for all} \quad n \in N_0.$$

If the series $\sum_{s=0}^{n} t_s$ *converges, then the zero solution of* (2) *is pth mean quasi-asymptotically stable.*

Proof. Set $F(n, u) = L_1 + L_2 u$, $u \in R^+$ for each $n \in N_0$. We show that condition (5c) in Theorem 5 holds. Let v_n be a solution of the difference equation $\Delta v_n = R t_{n+1}(L_1 + L_2 v_n)$. Then

$$v_n = v_0 \prod_{s=0}^{n-1} (1 + RL_2 t_{s+1}) + \sum_{s=0}^{n-1} RL_1 t_{s+1} \prod_{j=s+1}^{n-1} (1 + RL_2 t_{j+1}).$$

Since $\sum_{i=0}^{\infty} t_i < \infty$, by Lemma 1 there exists a positive constant B such that $v_n < B$ for all $n \in N_0$. Thus by Theorem 5 the zero solution of (2) is pth mean quasi-asymptotically stable. Therefore the proof of the corollary is complete.

We have the following corollary by Theorems 1 and 3.

COROLLARY 6. *Suppose that all conditions in Theorem* 3 *for the stochastic difference equation* (2) *are satisfied. Furthermore if the zero solution of the difference equation* $\Delta v_n = CF(n, v_n)$ ($F(n, 0) = 0$ *for all* $n \in N_0$) *is stable, then the zero solution of* (2) *is first mean asymptotically stable.*

DEFINITION 9. The solutions of (1) are said to approach zero fast with the order (K, p) if there exists a constant $K \geqslant (\frac{1}{2})^p$ such that

$$E(\|A_{m-1} \cdots A_n\|^p) \leqslant K/m^p \qquad \text{for all} \quad m > n \geqslant 0.$$

Finally we prove the following pth mean stability theorem of the nonlinear stochastic difference equation.

THEOREM 6. *Suppose that the following conditions are satisfied for the stochastic difference equation* (2):

(6a) *the solutions of* (1) *approach zero fast with the order* (K, p),

(6b) $E(\|f(n, x)\|^p) \leqslant F(n, E(\|x\|^p))$, $F(n, 0) = 0$ *for all* $n \in N_0$, *where* $x : \Omega \to R^n$ *is a random variable and* $F(n, u) : N_0 \times R^+ \to R^+$ *is a monotone non-decreasing continuous function in u for each fixed* $n \in N_0$,

(6c) *the zero solution of the difference equation*

$$\Delta v_n = 2^p K F(n, v_n)$$

is (*asymptotically*) *stable.*

Then the zero solution of (2) *is pth mean* (*asymptotically*) *stable.*

Proof. Since the solutions of (1) approach zero fast with the order (K, p), by (4) and condition (6b) we obtain that

$$E(\|x_n\|^p) \leqslant E\left(\left\|\prod_{j=0}^{n-1} A_j\right\| \|x_0\| + \sum_{s=0}^{n-1} \left\|\prod_{j=s+1}^{n-1} A_j\right\| \|f(s, x_s)\|\right)^p$$

$$\leqslant 2^p K E(\|x_0\|^p) + 2^p \sum_{s=0}^{n-1} n^p K (1/n)^p F(s, E(\|x_s\|^p))$$

$$= 2^p K E(\|x_0\|^p) + \sum_{s=0}^{n-1} 2^p K F(s, E(\|x_s\|^p)).$$

Hence by Lemma 3 and condition (6c) we obtain the conclusion of the theorem by the same way as in the proof of Theorem 2.

5. Stochastic Boundedness of Non-linear Difference Equations

In this section we consider the boundedness of solutions of the difference equation (2). First of all, let us define the stochastic boundedness of solutions of (1) and (2).

Definition 10. The solutions of (2) are said to be

(B_1) pth mean bounded for $p > 0$ if for any $\rho > 0$, there exists a $\beta(\rho) > 0$ such that

$$E(\|x_k\|^p) < \beta(\rho) \qquad \text{for all} \quad k \in N_0$$

provided $E(\|x_0\|^p) < \rho$,

(B_2) pth mean ultimately bounded for $p > 0$ if they are pth mean bounded, moreover there exists a real number $B > 0$ and for any $\rho > 0$, there exists a $T(\rho) > 0$ such that

$$E(\|x_k\|^p) < B \qquad \text{for all} \quad k \geqslant T(\rho)$$

provided $E(\|x_0\|^p) < \rho$.

Next we give the boundedness definition of solutions of the difference equation $\Delta v_n = G(n, v_n)$, where $G(n, u): N_0 \times R^+ \to R^+$.

Definition 11. The solutions of $\Delta v_n = G(n, v_n)$ are said to be

(BB_1) bounded if for any $\rho > 0$, there exists a $\beta(\rho) > 0$ such that

$$0 < v_k < \beta(\rho) \qquad \text{for all} \quad k \in N_0$$

provided $0 < v_0 < \rho$,

(BB_2) ultimately bounded if they are bounded, moreover there exists a real number $B > 0$ and for any $\rho > 0$, there exists a $T(\rho) > 0$ such that

$$0 < v_k < B \qquad \text{for all} \quad k \geqslant T(\rho)$$

provided $0 < v_0 < \rho$.

Now we prove stochastic boundedness theorems of (2).

THEOREM 7. *Suppose that conditions* (1a), (1b) *hold and moreover the following condition is satisfied for the stochastic difference equation* (2):

(7a) *the solutions of the difference equation*

$$\Delta v_n = CF(n, v_n)$$

are (*ultimately*) *bounded.*

Then the solutions of (2) *are first mean* (*ultimately*) *bounded.*

Proof. By the same way as in the proof of Theorem 1, we obtain that

$$E(\|x_n\|) \leqslant CE(\|x_0\|) + \sum_{s=0}^{n-1} CF(s, E(\|x_s\|)) \qquad \text{for all} \quad n \in N_0.$$

Let v_n be the solution of the difference equation $\Delta v_n = CF(n, v_n)$ with the initial value $v_0 = CE(\|x_0\|)$. Then applying Lemma 3 with $\rho(n) \equiv 1$, for all $n \in N_0$

$$E(\|x_n\|) \leqslant v_n,$$

which implies that the solutions of (2) are first mean (ultimately) bounded because of condition (7a).

COROLLARY 7. *Suppose that conditions* (1a), (1e) *hold for the stochastic difference equation* (2). *If the series* $\sum_{i=0}^{n} B(i)$ *converges, then the solutions of* (2) *are first mean bounded.*

THEOREM 8. *Suppose that conditions* (2a), (2b) (*or* (6a), (6b)) *hold and the following condition is satisfied for the stochastic difference equation* (2):

(8a) *the solutions of the difference equation*

$$\Delta v_n = 2^p KG(n, v_n) \qquad (\text{or} \qquad \Delta v_n = 2^p KF(n, v_n))$$

are (*ultimately*) *bounded.*

Then the solutions of (2) *are pth mean* (*ultimately*) *bounded.*

Proof. By the same way as in the proof of Theorem 2 (or Theorem 6) we obtain that

$$E(\|x_n\|^p) \leqslant 2^p K E(\|x_0\|^p) + \sum_{s=0}^{n-1} 2^p K G(s, E(\|x_s\|^p)) \qquad \text{for all} \quad n \in N_0$$

(or $E(\|x_n\|^p) \leqslant 2^p K E(\|x_0\|^p) + \sum_{s=0}^{n-1} 2^p K F(s, E(\|x_s\|^p))$ for all $n \in N_0$), from which it follows that the solutions of (2) are pth mean (ultimately) bounded because of condition (8a) and Lemma 3.

Finally we present ultimate boundedness theorems of the stochastic difference equation (2).

THEOREM 9. *Suppose that conditions* (3a), (3b) *hold and the following conditions are satisfied for the difference equation* (2):

(9a) *the solutions of* (2) *are first mean bounded,*

(9b) *there exists a* $B > 0$ *and for any* $\rho > 0$, *there exists a* $T(\rho) > 0$ *such that if* $v_0 < \rho$, *then* $\delta^n v_n < B$ *for* $n \geqslant T(\rho)$, *where* v_n *is a solution of the difference equation* $\Delta v_n = CF(n, v_n)$.

Then the solutions of (2) *are first mean ultimately bounded.*

THEOREM 10. *Suppose that conditions* (4a), (4b) *hold and the following conditions are satisfied for the difference equation* (2):

(10a) *the solutions of* (2) *are pth mean bounded,*

(10b) *there exists a* $B > 0$ *and for any* $\rho > 0$, *there exists a* $T(\rho) > 0$ *such that if* $v_0 < \rho$, *then* $\delta^n v_n < B$ *for* $n \geqslant T(\rho)$, *where* v_n *is a solution of the difference equation* $\Delta v_n = 2^p KF(n, v_n)$.

Then the solutions of (2) *are pth mean ultimately bounded.*

Proof. By the method similar to the proofs of Theorems 3 and 4, respectively, one could prove the above theorems. Thus we omit the proofs of the theorems.

Remark. If the solutions of $\Delta v_n = CF(n, v_n)$ and $\Delta v_n = 2^p KF(n, v_n)$ are bounded, respectively, conditions (9a), (9b) and conditions (10a), (10b) are satisfied, respectively.

REFERENCES

1. R. BELLMAN, T. T. SOONG, AND R. VASUDEVAN, On the moment behavior of a class of stochastic difference equations, *J. Math. Anal. Appl.* **40** (1972), 286–299.

2. L. Cesari, "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," Springer-Verlag, New York, 1973.
3. A. Friedman, "Stochastic Differential Equations and Applications," Vol. 1, Academic Press, New York, 1975.
4. S. R. Grace, B. S. Lalli, and C. C. Yeh, Comparison theorems for difference inequalities, *J. Math. Anal. Appl.* **113** (1986), 468–472.
5. G. S. Ladde and M. Sambandham, Random difference inequalities, *in* "Trends in the Theory and Practice of Non-Linear Analysis" (V. Lakshmikanham, Ed.), pp. 231–240, North-Holland, Amsterdam, 1985.
6. V. Lakshmikantham and S. Leela, "Differential and Integral Inequalities," Vol. 1, Academic Press, New York, 1969.
7. V. Lakshmikantham and D. Trigiante, "Theory of Difference Equations," Academic Press, San Diego, 1988.
8. F. Ma and T. K. Caughey, On the stability of stochastic difference systems, *Internat. J. Non-Linear Mech.* **16** (1981), 139–153.
9. F. Ma and T. K. Caughey, On the stability of linear and nonlinear stochastic transformations, *Internat. J. Control* **34** (1981), 501–511.
10. F. Ma and T. K. Caughey, Mean stability of stochastic difference systems, *Internat. J. Non-Linear Mech.* **17** (1982), 69–84.
11. F. Ma, Stability theory of stochastic difference systems, *in* "Probabilistic Analysis and Related Topics" (A. T. Bharucha-Reid, Ed.), Vol. 3, pp. 127–160, Academic Press, New York/London, 1983.
12. J. M. Ortega, Stability of difference equations and convergence of iterative processes, *SIAM J. Numer. Anal.* **10** (1973), 268–282.
13. B. B. Pachpatte, Finite difference inequalities and an extension of Liapunov method, *Michigan Math. J.* **18** (1971), 385–391.
14. R. Redhefter and W. Walter, A comparison theorem for differential inequalities, *J. Differential Equations* **44** (1982), 111–117.
15. S. Sugiyama, "Difference Inequalities and Their Applications to Stability Problems," Lecture Notes in Mathematics, Vol. 243, pp. 1–15, Springer-Verlag, New York/Berlin, 1971.
16. E. C. Titchmarsh, "The Theory of Functions," Oxford Univ. Press, London, 1939.